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Entanglement production and decoherence-free subspace of two single-mode cavities embedded in a common environment

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Abstract

A system consisting of two identical single-mode cavities coupled to a common environment is investigated within the framework of algebraic dynamics. Based on the left and right representations of the Heisenberg–Weyl algebra, the algebraic structure of the master equation is explored and exact analytical solutions of this system are obtained. It is shown that for such a system, the environment can produce entanglement in contrast to its commonly believed role of destroying entanglement. In addition, the collective zero-mode eigensolutions of the system are found to be free of decoherence against the dissipation of the environment. These decoherence-free states may be useful in quantum information and quantum computation.

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1. Introduction

Quantum entanglement plays a basic role in quantum communication and quantum computation [1]. The creation of entanglement between qubits is of fundamental importance for further quantum computation processing. The entanglement can be created by a direct interaction between qubits [2] or an indirect interaction via a third party [3]. However, both of the above processes are confined in closed system, i.e., the influences of environment are neglected. Recent investigations showed that environment can be helpful to the entanglement creation in an open system [4, 5], which provide a perspective to use the environment to implement decoherence-free quantum information processing [6, 7]. In order to treat the influences of the environment on an open quantum system, the Born–Markovian master equation approach has been widely used. A common feature of the quantum master equations

is the existence of the sandwich terms of the Liouville operator where the reduced density matrix of the system is in between some quantum excitation and de-excitation operators. These terms result from the elimination of an enormous irrelevant degree of freedom of the environment. Except for some simple cases, for example, a single mode of the cavity field coupled to the vacuum or stationary regime properties [8], it is very difficult to solve directly the master equation. Instead, it is usual to convert the master equations into some c-number equations in the coherent state representation—the Fokker–Planck equation [9, 10].

In the previous works [11–13], we have proposed and developed an algebraic method to treat the sandwich terms in the Liouville operator for quantum statistical systems. This method is just a generalization of the algebraic dynamical method [14] from quantum mechanical systems to quantum statistical systems. According to the characteristic of the sandwich terms in the Liouville operator, the left and right representations of the relevant algebra [15] have been introduced and the corresponding composite algebra has been constructed. As a result, the master equation has been converted into a Schrödinger-like equation and the problems can be solved exactly.

In this paper, we shall use this method to solve the problem of two identical cavities coupled to a common environment. The system consisting of two coupled cavities and a similar system of two coupled harmonic oscillators are important in quantum optics and quantum information theory. From these systems, many properties, such as information transfer of quantum states [16], and the quantum statistical properties of the two-coupled modes of electromagnetic fields [17] have been investigated. In [18], the entanglement of the two-coupled harmonic oscillator systems has been quantitatively studied. However, all these studies neglect the effect of environment on the modes of cavities and the system is thus closed. Although [19] studied the dissipative two oscillator system, it was based on the quantum characteristic function approach and the direct analytical solutions are not obtained. In this work, we shall consider a specific coupling of the two cavity fields, induced through the individual interactions of the two cavities with the environment and our system is thus different from that of [19]. By introducing a collective mode consisting of the two cavity modes, the quantum master equation is obtained. The two independent cavities are thus interacting with each other indirectly through the environment. Using the algebraic dynamical method, the full algebraic structure of the master equation is explored and its exact analytical solution is obtained. From the zero eigenvalue solutions, the decoherence-free sub-space is obtained. Finally, the exact entanglement dynamics for the single- and two-photon processes is obtained from the time-dependent solutions. We find that the system exhibits an interesting feature of environment-assisted entanglement generation. Our present study shows that the environment is not just a negative source of decoherence, it can also be a positive source of entanglement.

This paper is organized as follows. In section 2, the model Hamiltonian of the system is presented and the master equation for the reduced density matrix of the system is obtained. In section 3, based on the left and right representations of the relevant algebra, we introduce a composite algebra, in terms of which the dynamical $su(1, 1)$ plus $u(2)$ algebraic structures of the Liouville operator (rate operator) of the master equation are found. Section 4 is devoted to the analytical solutions of the master equation. The applications of its time-dependent solutions to the entanglement dynamics for the single- and two-photon processes are studied in section 5. Finally, a brief summary is given in section 6.

2. Two identical cavities in a common environment

Consider two identical cavities (or two harmonic oscillators) interacting with a common environment. With the dipole interaction and in the rotating wave approximation, the system

can be described by the Hamiltonian

$$H = \omega \sum_i a_i^\dagger a_i + \sum_k \omega_k b_k^\dagger b_k + \sum_{i,k} (g_{ik} a_i^\dagger b_k + \text{h.c.}),$$

where a_i ($i = 1, 2$) are boson operators of the two cavities with the same frequency ω and b_k are those for the environment fields, respectively. g_{ik} is the coupling constant. It is reasonable to assume that the coupling of the two cavities to the environment is same, i.e., $g_{ik} = g_k$. This is a generation of Dicke limit of quantum optics [20]. Using the standard technique of quantum optics, one can obtain the master equation for the reduced density of the two-cavity system under the standard Born–Markovian approximation [8, 10]

$$\begin{aligned} \frac{d\rho(t)}{dt} = & \frac{\zeta}{2} \{ [2a_1\rho(t)a_1^\dagger - a_1^\dagger a_1\rho(t) - \rho(t)a_1^\dagger a_1] + [2a_2\rho(t)a_2^\dagger - a_2^\dagger a_2\rho(t) - \rho(t)a_2^\dagger a_2] \} \\ & + \frac{\zeta}{2} \{ [2a_1\rho(t)a_2^\dagger - a_1^\dagger a_2\rho(t) - \rho(t)a_1^\dagger a_2] \\ & + [2a_2\rho(t)a_1^\dagger - a_2^\dagger a_1\rho(t) - \rho(t)a_2^\dagger a_1] \}, \end{aligned} \quad (1)$$

where ζ (> 0) is the decay constant of the collective mode. It is noted that on the right-hand side of equation (1) the first two terms denote the individual dissipation of the two cavities due to the environment, while the last two terms describe the coupling (photon exchange) between the two cavities mediated by the environment. Equation (1) is complicated and can be simplified by introducing a collective (quasi-photon) mode operator $A = \frac{1}{\sqrt{2}}(a_1 + a_2)$, as did in [21]. In doing so, the master equation of the reduced density becomes

$$\frac{d\rho(t)}{dt} = \frac{\zeta}{2} [2A\rho(t)A^\dagger - A^\dagger A\rho(t) - \rho(t)A^\dagger A]. \quad (2)$$

In the following we shall discuss the algebraic structure of this system.

3. Algebraic structure of the master equation

Based on the left and right representations of Heisenberg–Weyl algebra [12], the master equation (2) can be converted into a Schrödinger-like equation and the algebraic structure can thus be established. Noting the fact that the collective operators A and A^\dagger obey the same commutation relations as the photon operators a and a^\dagger do, $\{A, A^\dagger, A^\dagger A, 1\}$ thus constitutes a $hw(4)$ algebra. Define the left and right algebras of $hw(4)$

$$\begin{aligned} hw(4)_r &= \{A^r, A^{r\dagger}, \tilde{N}^r = A^{r\dagger} A^r, 1\}, \\ hw(4)_l &= \{A^l, A^{l\dagger}, \tilde{N}^l = A^{l\dagger} A^l, 1\}, \end{aligned} \quad (3)$$

where $hw(4)_r$ acts to the right on the ket photon number state $|n\rangle$ and $hw(4)_l$ acts to the left on the bra photon number state $\langle n|$. They have the following commutators:

$$\begin{aligned} [A^r, A^{r\dagger}] &= 1, & [A^r, \tilde{N}^r] &= A^r, & [A^{r\dagger}, \tilde{N}^r] &= -A^{r\dagger}; \\ [A^l, A^{l\dagger}] &= -1, & [A^l, \tilde{N}^l] &= -A^l, & [A^{l\dagger}, \tilde{N}^l] &= A^{l\dagger}. \end{aligned} \quad (4)$$

It is noted that $hw(4)_l$ ($hw(4)_r$) is isomorphic (anti-isomorphic) to $hw(4)$. Since $hw(4)_l$ and $hw(4)_r$ act on different spaces, bra and ket spaces, the operators commute with each other, i.e., $[hw(4)_l, hw(4)_r] = 0$. The reduced density matrix is a vector of the von Neumann super space, which has the $hw(4)_l \otimes hw(4)_r$ algebraic structure containing $sp(4)$ as a relevant sub-algebra, as shown below.

From the above basic algebras we can constitute a composite algebra C in the adjoint representation,

$$C = \{K_- = A^r A^{l\dagger}, K_+ = A^{r\dagger} A^l, K_0 = \frac{1}{2}(\tilde{N}^r + \tilde{N}^l)\}.$$

We see that C is an $su(1, 1)$ algebra satisfying the following commutation rules,

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_-, K_+] = 2K_0,$$

which is a sub-algebra of the $sp(4)$ and can be derived from equations (4). The operator K_- acts on the photon number bases of the von Neumann space as follows:

$$K_-|n_1 n_2\rangle\langle m_1 m_2| = \frac{1}{2}(\sqrt{n_1}|n_1 - 1, n_2\rangle + \sqrt{n_2}|n_1, n_2 - 1\rangle) \\ \times (\langle m_1 - 1, m_2|\sqrt{m_1} + \langle m_1, m_2 - 1|\sqrt{m_2}). \quad (5)$$

Similarly, for the super-vector bases of the quasi-photon (collective photon) number states $|n\rangle_{qq}\langle m|$ marked by the subscript q , the actions of the de-excitation operator K_- and the number operator K_0 read

$$K_-|n\rangle_{qq}\langle m| = \sqrt{nm}|n - 1\rangle_{qq}\langle m - 1|, \\ K_0|n\rangle_{qq}\langle m| = \frac{n + m + 1}{2}|n\rangle_{qq}\langle m|. \quad (6)$$

With this composite algebra at hand, it is straightforward to convert the master equation (2) into the Schrödinger-like equation

$$\frac{d\rho(t)}{dt} = \Gamma\rho(t), \quad \Gamma = \varsigma K_- - \varsigma K_0 + \frac{\varsigma}{2}. \quad (7)$$

Since the rate operator Γ is a linear function of the $su(1, 1)$ generators, we conclude that the master equation possesses an $su(1, 1)$ dynamical symmetry in the quasi-photon number representation. Thus the system is integrable and can be solved analytically according to the algebraic dynamics [14].

It is remarkable to note that K_0 is diagonalized in the quasi-photon representation, but it is not so in the real photon number representation of the von Neumann space of the density matrix. As will be seen soon, it has a different algebraic structure in the real photon number representation. In terms of the two real photon operators, K_0 can be written as

$$K_0 = \frac{1}{4}(N_1 + N_2 + S_+ + S_-),$$

where $N_i = a_i^{r\dagger} a_i^r + a_i^{l\dagger} a_i^l = n_i^r + n_i^l$ ($i = 1, 2$), $S_+ = a_1^{r\dagger} a_2^r + a_1^l a_2^{l\dagger}$ and $S_- = a_1^r a_2^{r\dagger} + a_1^l a_2^l$. It is not difficult to find that the operators (N_1, N_2, S_+, S_-) form a $u(2)$ ($= u(1) + su(2)$) algebra

$$u(2) = \{S_0 = (N_1 - N_2)/2, S_+, S_-; N = (N_1 + N_2)/2\},$$

which obey the following commutation rules:

$$[N, S_{\pm}] = [N, S_0] = 0, \\ [S_0, S_{\pm}] = \pm S_{\pm}, [S_-, S_+] = -2S_0.$$

These operators act on the von Neumann space in the real photon number representation as follows:

$$S_+|n_1 n_2\rangle\langle m_1 m_2| = \sqrt{(n_1 + 1)n_2}|n_1 + 1, n_2 - 1\rangle\langle m_1 m_2| + \sqrt{(m_1 + 1)m_2}|n_1 n_2\rangle\langle m_1 + 1, m_2 - 1|, \\ S_-|n_1 n_2\rangle\langle m_1 m_2| = \sqrt{n_1(n_2 + 1)}|n_1 - 1, n_2 + 1\rangle\langle m_1 m_2| + \sqrt{m_1(m_2 + 1)}|n_1 n_2\rangle\langle m_1 - 1, m_2 + 1|, \\ N_i|n_1 n_2\rangle\langle m_1 m_2| = (n_i + m_i + 1)|n_1 n_2\rangle\langle m_1 m_2| \quad (i = 1, 2), \\ S_0|n_1 n_2\rangle\langle m_1 m_2| = (n_1 + m_1 - n_2 - m_2)/2|n_1 n_2\rangle\langle m_1 m_2|, \\ N|n_1 n_2\rangle\langle m_1 m_2| = (n_1 + m_1 + n_2 + m_2 + 2)/2|n_1 n_2\rangle\langle m_1 m_2|. \quad (8)$$

Up to now we have explored the complete dynamical symmetry of the system and the results can be summarized as follows: the largest dynamical algebra of the two-cavity system

is the $hw(4)_l \otimes hw(4)_r \supset sp(4)$ algebra since it has two kinds of photons; due to the special structure of the rate operator Γ , the system in fact has the $su(1, 1)$ and $u(2)$ sub-algebraic dynamical symmetries of the largest dynamical symmetry algebra $sp(4)$. For the quasi-photon, the dynamical symmetry is of the $su(1, 1)$ sub-algebra and for the two kinds of real photons it is $u(2)$ algebra.

4. Solutions of the master equation

4.1. Eigensolution and decoherence-free states

To better understand the property of steady solution of the master equation, we first investigate the eigensolutions of the master equation. The eigenequation reads

$$\Gamma\rho = \gamma\rho. \quad (9)$$

To solve equation (9), we introduce two similarity transformations corresponding to the above two sub-algebraic structures. The first one is

$$\rho = U_1\rho_1, \quad U_1 = e^{-K_-}.$$

The operator Γ is thus diagonalized in the quasi-photon number representation as follows:

$$\bar{\Gamma}_1 = U_1^{-1}\Gamma U_1 = -\zeta K_0 + \frac{\zeta}{2}. \quad (10)$$

From equation (10) the zero-mode eigensolution of Γ in the quasi-photon number representation can be obtained as

$$\rho_0 = |0\rangle_{qq}\langle 0|, \quad (11)$$

which has the same form as the stationary solution of the master equation for the single-mode damped harmonic oscillator [8].

One notes that in the quasi-photon number representation, the collective quasi-photon mode is used, in which two cavities are coupled. In experiments, it is convenient to use the real photon number representation. Moreover, in this representation, the interaction of the two cavities can also be studied.

To diagonalize the rate operator $\bar{\Gamma}_1$ (or K_0) in the real photon number representation, the second similarity transformation is needed, namely,

$$\bar{\Gamma}_2 = U_2^{-1}\bar{\Gamma}_1 U_2, \quad (12)$$

where U_2 is defined as

$$\rho_1 = U_2\rho_2, \quad U_2 = e^{\beta_+ S_+} e^{\beta_- S_-}.$$

Under the conditions

$$1 - \beta_+^2 = 0, \quad 2\beta_+\beta_- + 1 = 0, \quad (13)$$

i.e., $\beta_+ = \pm 1$ and $\beta_- = \mp \frac{1}{2}$, the rate operator $\bar{\Gamma}_2$ is diagonalized in terms of both of the two Cartan operators N and S_0 of the $u(2)$ algebra and the real photon number representation (N_1, N_2) ,

$$\begin{aligned} \bar{\Gamma}_2 &= \frac{-\zeta}{2}(N - \beta_+ S_0) + \frac{\zeta}{2}, \\ &= \frac{-\zeta}{4}(f_2 N_1 + f_3 N_2) + \frac{\zeta}{2}, \\ f_2 &= 1 - \beta_+, \\ f_3 &= 1 + \beta_+. \end{aligned}$$

From the expression of $\bar{\Gamma}_2$ we can see that the eigensolutions of $\bar{\Gamma}_2$ are highly degenerate with respect to different photon number distributions among the bra and ket states. So besides the two Cartan operators we need two additional quantum numbers to distinguish the degenerate states. Then the complete set of commutation quantum operators is $\{N, S_0, n_1^r, n_2^r\}$. For convenience, we choose the equivalent set of the complete commutation quantum operators as $\{n_1^r, n_1^l, n_2^r, n_2^l\}$.

After making two inverse transformations of $\bar{\Gamma}_2$ and $\bar{\Gamma}_1$, we obtain the eigensolution of equation (9) on the real photon number bases of the von Neumann space as follows,

$$\begin{aligned}\gamma(n_1 n_2, m_1 m_2) &= \frac{-\zeta}{4} [f_2(n_1 + m_1 + 1) + f_3(n_2 + m_2 + 1)] + \frac{\zeta}{2}, \\ \rho(n_1 n_2, m_1 m_2) &= e^{-K_-} e^{\beta_+ S_+} e^{\beta_- S_-} |n_1 n_2\rangle \langle m_1 m_2|.\end{aligned}\quad (14)$$

Substituting the two sets of solutions of equations (13) into the expression of γ , we see that both of them contain the quasi-photon zero-mode solution. For the second similarity transformation, the two sets of solutions of equations (13) are physically equivalent and degenerate. This can be seen from the expressions of γ , f_2 and f_3 in terms of the transformation coefficient β_+ which produces the solutions labelled by n_1 and m_1 (or n_2 and m_2), since f_2 and f_3 cannot be non-zero simultaneously under the similarity transformation. The degeneracy of the zero quasi-photon mode comes from the different real photon number distributions among the two cavities. From equations (14) we can express the zero quasi-photon mode solution of the system in terms of the real photon number bases as follows,

$$\rho_0 = c_0 e^{-K_-} e^{S_+} e^{-\frac{1}{2} S_-} |n0\rangle \langle m0| = c'_0 e^{-K_-} e^{-S_+} e^{\frac{1}{2} S_-} |0n\rangle \langle 0m| \quad (15)$$

where c_0 and c'_0 are normalized constants. After some straightforward calculations, it is found that the action of e^{-K_-} has no effect on the states $e^{S_+} e^{-\frac{1}{2} S_-} |n0\rangle \langle m0|$ (or $e^{-S_+} e^{\frac{1}{2} S_-} |0n\rangle \langle 0m|$). Since $\Gamma \rho_0 = 0$, ρ_0 is an invariant subspace under the time evolution of the master equation (7). It is found that the quasi-photon zero-mode subspace (15) contains many entangled real photon states in the two cavities, which are thus decoherence-free with respect to the action of the dissipative operator Γ in the master equation (7). For example, for the zero-photon process, the zero-mode state is $\rho_0^{(0)} = |00\rangle \langle 00|$; for the single-photon process, the zero-mode state is $\rho_0^{(1)} = |\phi\rangle \langle \phi|$, where $|\phi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$; for the two-photon process, the zero-mode state is $\rho_0^{(2)} = |\varphi\rangle \langle \varphi|$, where $|\varphi\rangle = \frac{1}{2}(|02\rangle + |20\rangle - \sqrt{2}|11\rangle)$, etc. All the above entangled real photon states are the members of the quasi-photon zero-mode subspace and decoherence-free from the dissipative action of Γ in the master equation (7).

From the above analysis, we see that the quasi-photon zero-mode solution (15) forms a highly degenerate and dissipation-free collective subspace in terms of different real photon number states which are orthogonal to each other, and many of them are entangled and thus decoherence-free. It is noted that the equilibrium steady solutions of the system depending on initial states are not unique, as shown later. All of them are within the quasi-photon zero-mode subspace and consisting of all possible mixtures of these real photon states with collective zero mode.

4.2. Time-dependent solutions of the master equation

Next we shall investigate the dynamical properties of the system based on the real photon number representation. First we solve the master equation (7) and get its time-dependent solutions. Then we analyse the time evolution behaviour of the solutions and prove that for some initial product states the entanglement can be produced by the environment induced dynamics and for some initial state the system is decoherence-free.

By introducing a time-dependent gauge transformation

$$\rho = U_1(t)\tilde{\rho}, \quad U_1(t) = e^{\alpha_-(t)K^-},$$

the master equation can be rewritten as a diagonal form in the quasi-photon number representation

$$\frac{d\tilde{\rho}(t)}{dt} = \tilde{\Gamma}'\tilde{\rho}(t), \quad \tilde{\Gamma}' = U_1^{-1}(t)\Gamma U_1(t) - U_1^{-1}(t)\frac{dU_1(t)}{dt} = -\zeta K_0 + \frac{\zeta}{2},$$

if the following gauge transformation conditions are satisfied:

$$\frac{d\alpha_-(t)}{dt} = \zeta[1 + \alpha_-(t)]. \tag{16}$$

The time-dependent solution is now

$$\begin{aligned} \rho(t) &= U_1(t)\tilde{\rho}(t) = U_1(t) e^{\int_0^t \tilde{\Gamma}'(\tau) d\tau} \rho(0) \\ &= \sum_{m,n} C_{m,n} e^{\alpha_-(t)K^-} e^{-\frac{\zeta(n+m)t}{2}} |n\rangle_{qq} \langle m|, \end{aligned} \tag{17}$$

where we have used the initial super state as follows $\rho(0) = \tilde{\rho}(0) = \sum_{m,n} C_{m,n} |n\rangle_{qq} \langle m|$.

Similar to the steady case, it is necessary to express the above solution in the real-photon number representation. To this end, one needs the second time-dependent gauge transformation

$$U_2(t) = e^{\beta_+(t)S_+} e^{\beta_-(t)S_-},$$

where the initial condition is taken as $U_2(0) = 1$. Under the diagonalization conditions

$$\frac{d\beta_+(t)}{dt} = \frac{-\zeta}{4}(1 - \beta_+(t)^2), \quad \frac{d\beta_-(t)}{dt} = \frac{-\zeta}{4}(1 + 2\beta_+(t)\beta_-(t)), \tag{18}$$

the rate operator $\tilde{\Gamma}'(t)$ (thus $\Gamma(t)$) can be diagonalized in the real photon representation and reads

$$\tilde{\Gamma} = U_2^{-1}(t)\tilde{\Gamma}'U_2(t) - U_2^{-1}(t)\frac{dU_2(t)}{dt} = \frac{-\zeta}{2}[N - \beta_+(t)S_0] + \frac{\zeta}{2}.$$

After the two inverse transformations of $U_1(t)$ and $U_2(t)$, the time-dependent solution of equation (1) is then

$$\begin{aligned} \rho(t) &= U_1(t)U_2(t)\tilde{\rho}(t) \\ &= U_1(t)U_2(t) \sum_{\substack{m_1, n_1 \\ n_2, m_2}} C_{n_1, n_2, m_1, m_2} \exp\left(\int_0^t \left\{ \frac{-\zeta}{4}[(1 - \beta_+(\tau))(n_1 + m_1 + 1) \right. \right. \\ &\quad \left. \left. + (1 + \beta_+(\tau))(n_2 + m_2 + 1)] + \frac{\zeta}{2} \right\} d\tau\right) |n_1 n_2\rangle \langle m_1 m_2|, \end{aligned} \tag{19}$$

where we have set the initial state of the system as $\rho(0) = \tilde{\rho}(0) = \sum_{m_1, n_1, n_2, m_2} C_{n_1, n_2, m_1, m_2} |n_1 n_2\rangle \langle m_1 m_2|$ in the real photon number representation. This solution is our central result in the present work. In the following we consider an explicit initial state to study explicitly the properties of this solution.

5. Application of the solution

In this section, we first consider explicitly the applications of the solution obtained above to the single- and two-photon processes, which show some interesting properties. Then we give a brief discussion on the implication to the quantum communication and quantum computation.

5.1. Single-photon process

Consider an initial state as $\rho(0) = |\chi\rangle\langle\chi|$, where $|\chi\rangle = a|01\rangle + \sqrt{1-a^2}|10\rangle$, and a denotes the relevant amplitude. In terms of equations (5), (8), the time-dependent solution (19) can be written as

$$\rho(t) = \rho_{01,01}(t)|01\rangle\langle 01| + \rho_{10,10}(t)|10\rangle\langle 10| + \rho_{\text{off}}(t)[|10\rangle\langle 01| + |01\rangle\langle 10|] + \rho_{00,00}(t)|00\rangle\langle 00|, \quad (20)$$

where the time-dependent coefficients can be obtained analytically,

$$\begin{aligned} \rho_{01,01}(t) &= a^2 \exp\left(\int_0^t -\frac{\zeta(\beta_+(\tau)+1)}{2} d\tau\right) + (1-a^2)\beta_-(t)^2 \exp\left(\int_0^t \frac{\zeta(\beta_+(\tau)-1)}{2} d\tau\right) \\ &\quad + 2a\sqrt{1-a^2}\beta_-(t) e^{-\frac{\zeta t}{2}}, \\ \rho_{10,10}(t) &= a^2\beta_+(t)^2 \exp\left(\int_0^t -\frac{\zeta(\beta_+(\tau)+1)}{2} d\tau\right) + (1-a^2)[1+\beta_+(t)\beta_-(t)]^2 \\ &\quad \times \exp\left(\int_0^t \frac{\zeta(\beta_+(\tau)-1)}{2} d\tau\right) + 2a\sqrt{1-a^2}\beta_+(t)[1+\beta_+(t)\beta_-(t)] e^{-\frac{\zeta t}{2}}, \\ \rho_{\text{off}}(t) &= a^2\beta_+(t) \exp\left(\int_0^t -\frac{\zeta(\beta_+(\tau)+1)}{2} d\tau\right) + (1-a^2)[1+\beta_+(t)\beta_-(t)]\beta_-(t) \\ &\quad \times \exp\left(\int_0^t \frac{\zeta(\beta_+(\tau)-1)}{2} d\tau\right) + a\sqrt{1-a^2}[1+2\beta_+(t)\beta_-(t)] e^{-\frac{\zeta t}{2}}, \\ \rho_{00,00}(t) &= \frac{\alpha_-(t)}{2} \left\{ a^2[1+\beta_+(t)]^2 \exp\left(\int_0^t -\frac{\zeta(\beta_+(\tau)+1)}{2} d\tau\right) + (1-a^2)[1+\beta_-(t) \right. \\ &\quad \left. + \beta_+(t)\beta_-(t)]^2 \exp\left(\int_0^t \frac{\zeta(\beta_+(\tau)-1)}{2} d\tau\right) \right. \\ &\quad \left. + 2a\sqrt{1-a^2}[1+\beta_+(t)][1+\beta_-(t)+\beta_+(t)\beta_-(t)] e^{-\frac{\zeta t}{2}} \right\}. \quad (21) \end{aligned}$$

From the above solution we can see that $U_2(t)$ makes the re-distribution of the photon number between the two cavities, which produces entanglement between them, while $U_1(t)$ induces de-excitation of the collective modes and leads to the zero collective mode. In order to study explicitly the time evolution of the system, it is needed to solve the time-dependent transformation parameters $\alpha_-(t)$ and $\beta_{\pm}(t)$, which can be obtained from equations (16) and (18),

$$\alpha_-(t) = e^{\zeta t} - 1, \quad \beta_+(t) = -\tanh\left(\frac{\zeta t}{4}\right), \quad \beta_-(t) = -\frac{1}{2} \sinh\left(\frac{\zeta t}{2}\right). \quad (22)$$

It is known that the entanglement between the two subsystems can be measured by a quantity named as the entanglement of formation [22]. This quantity can also be calculated by concurrence $C(\rho)$ [23]. As usual, $C(\rho)$ is defined as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4),$$

where λ_i ($i = 1, \dots, 4$) are eigenvalues of matrix

$$[\rho^{1/2}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)\rho^{1/2}]^{1/2}$$

and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Here $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and ρ^* is the complex conjugation of ρ relative to the eigenbasis of σ_z . It can be shown that the concurrence varies from $C = 0$ for a disentangled

state to $C = 1$ for a maximally entangled state. According to the solution (21) it is easy to find that the concurrence $C(\rho) = 2|\rho_{\text{off}}(t)|$ in the present case.

In the following we will consider three cases of the entanglement and decoherence dynamics. For the first case we show that, for product state, the steady entanglement between the two subsystems can be produced by the environment. In the second case we present the decoherence-free state which is stable against the dissipation. The last case represents the situation where the environment plays the conventional role on system, i.e. it induces decoherence and makes the system disentangled.

Case 1, $a = 0$. In this case the initial state is a product state and has no entanglement. Substituting equations (22) into equation (21) we have

$$\rho(t) = \frac{(1 - e^{-\frac{\gamma t}{2}})^2}{4} |01\rangle\langle 01| + \frac{(1 + e^{-\frac{\gamma t}{2}})^2}{4} |10\rangle\langle 10| + \frac{e^{-\gamma t} - 1}{4} [|01\rangle\langle 10| + |10\rangle\langle 01|] + \frac{1 - e^{-\gamma t}}{2} |00\rangle\langle 00|.$$

After a long time evolution, it is noted that all time-dependent coefficients become constant, and the density matrix can be denoted as

$$\rho_s = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\rho_0^{(0)} + \frac{1}{2}\rho_0^{(1)}, \quad (23)$$

$$\rho_0^{(0)} = |00\rangle\langle 00|, \quad \rho_0^{(1)} = |\phi\rangle\langle \phi|$$

which is in the basis of $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and $\rho_0^{(i)}$ ($i = 0, 1$) are obtained in section 4.1 as zero-mode eigensolutions for zero- and single-photon processes, respectively. One can evaluate the concurrence of this steady state as $C(\rho) = 0.5$. The steady state is the possible mixture of the degenerate zero-mode eigensolutions. This is consistent with the discussion in section 4.1. From the purification scheme proposed in [22] one can get the maximal entangled state $|\phi\rangle$ from a collective pair of the states above with probability $1/16$. Thus it is shown that the maximal entangled state $|\phi\rangle$ between the two cavities can be produced from a product state by the environment.

The time evolution behaviour of the density matrix and the concurrence is shown in figures 1 and 2, respectively.

The same discussion can be applied to the initial product states with $a = \pm 1$, in which $\rho(t)$ and $C(\rho)$ have the similar time evolution behaviour.

Case 2, $a = -1/\sqrt{2}$. The initial state is the maximally entangled state. In this case we found that the density matrix is time-independent, i.e.,

$$\rho(t) = \frac{1}{4}|01\rangle\langle 01| + \frac{1}{4}|10\rangle\langle 10| - \frac{1}{4}[|01\rangle\langle 10| + |10\rangle\langle 01|] = |\phi\rangle\langle \phi|.$$

Thus, this solution is decoherence-free and the concurrence is also a maximal constant $C(\rho) = 1$. From section 4.1 we know that $|\phi\rangle\langle \phi|$ is the zero-mode eigen solution for the single-photon process of the master equation (1).

Case 3, $a = 1/\sqrt{2}$. In this case we obtain the time-dependent solution as follows:

$$\rho(t) = \frac{e^{-\gamma t}}{2} \{|01\rangle\langle 01| + |10\rangle\langle 10| + [|01\rangle\langle 10| + |10\rangle\langle 01|]\} + (1 - e^{-\gamma t})|00\rangle\langle 00|.$$

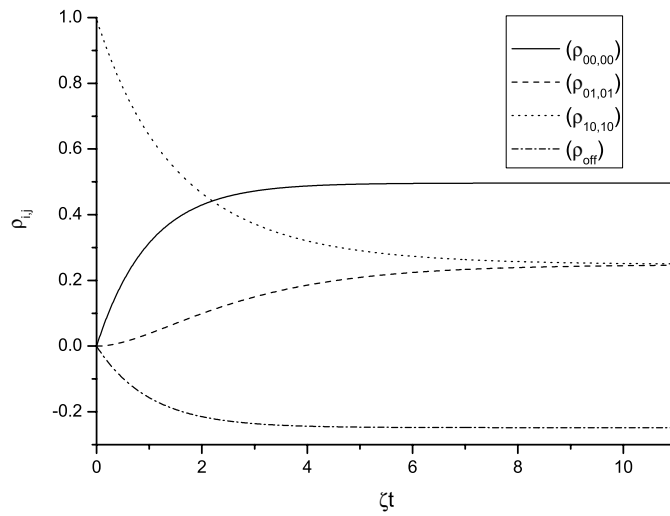


Figure 1. The time evolution of the elements of density matrix with the initial condition of $a = 0$.

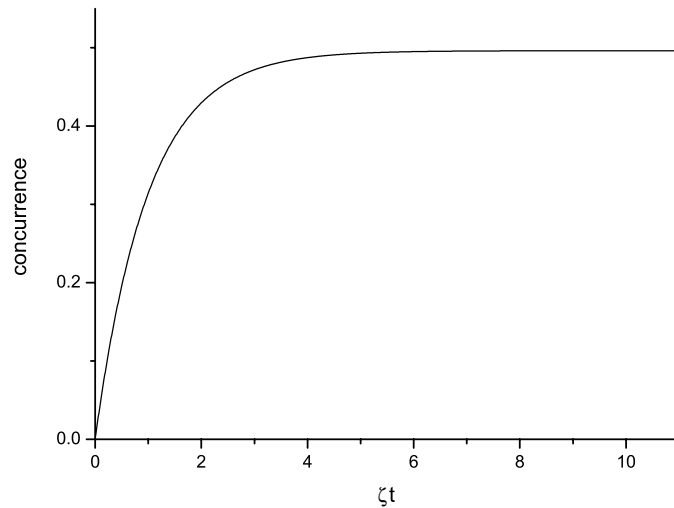


Figure 2. The time evolution of the concurrence $C(\rho)$ with the initial condition of $a = 0$.

Its time evolution is plotted in figure 3, and asymptotically the state approach to a product state with zero real photon number.

$$\rho_s = \rho_0^{(0)}.$$

The concurrence $C(\rho)$ decays quickly, as shown in figure 4, indicating that the decoherence is very strong and the two cavities will lose all of the information contained in its entangled initial state.

Except for the above three special cases, it is difficult to obtain a simple expression of the density matrix for general cases of a and numerical calculations are needed. In figure 5, we show the time evolution of the density matrix for a general initial state characterized by a . Figure 6 presents the time behaviour of the corresponding concurrence.

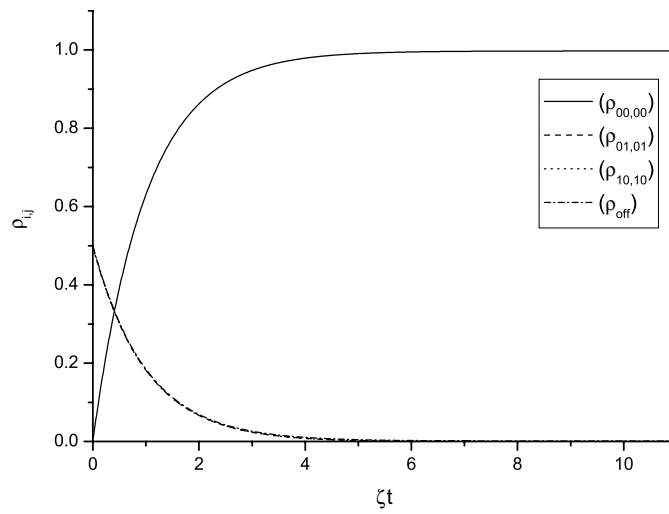


Figure 3. The time evolution of the elements of density matrix with the initial condition of $a = 1/\sqrt{2}$.

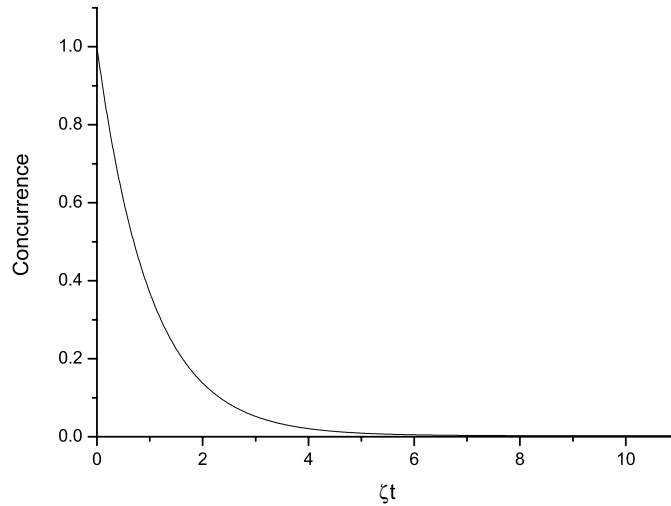


Figure 4. The time evolution of the concurrence $C(\rho)$ with the initial condition of $a = 1/\sqrt{2}$.

5.2. Two-photon process

In this two-photon process, each mode of the two cavities is related to three states, $|0\rangle$, $|1\rangle$ and $|2\rangle$, i.e., a qutrit. Therefore, this system is a potential candidate for a qutrit quantum information processing [24, 25]. The advantage of qutrits instead of qubits is that the qutrits are much more secure against the symmetric attacks in a quantum key distribution protocol, as shown by Bruss and Macchiavello [25]. In a recent experiment [26], the arbitrary qutrit states have been realized on the single-mode biphoton field, which shares some similarities with our model considered here. Based on the exact solution (19) in the two-photon process we shall analyse the decoherence effects on the qutrit state generation.

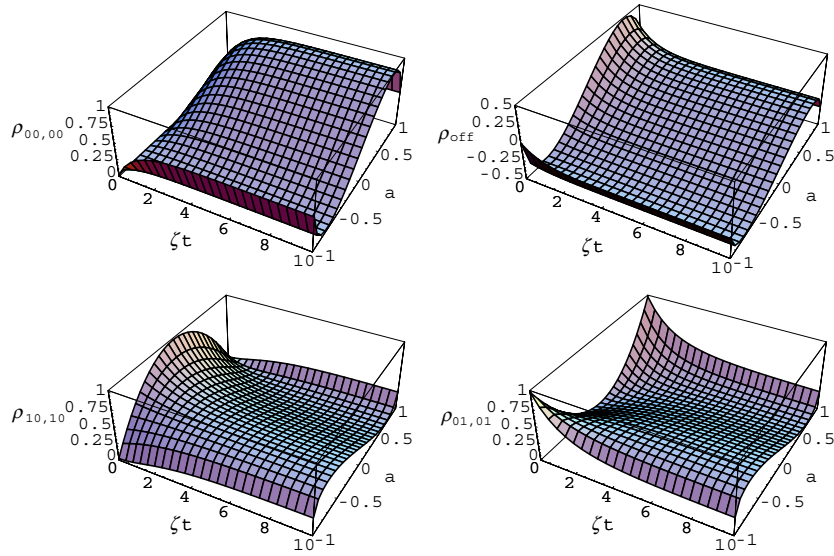


Figure 5. The time evolution of the elements of density matrix with a general initial condition given by a .

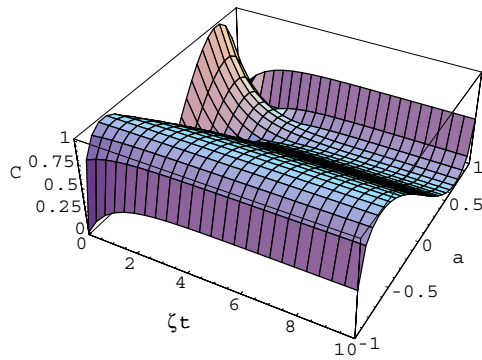


Figure 6. The time evolution of the concurrence $C(\rho)$ with different a .

Following [27], the entanglement of bipartite qutrit states can be measured in terms of the Schmidt coefficients by

$$\begin{aligned}
 |\Phi_{3 \times 3}\rangle &= k_1|x_1, y_1\rangle + k_2|x_2, y_2\rangle + k_3|x_3, y_3\rangle, \\
 C(|\Phi_{3 \times 3}\rangle) &= \sqrt{3(k_1^2k_2^2 + k_1^2k_3^2 + k_2^2k_3^2)},
 \end{aligned}
 \tag{24}$$

where equation (24) is a generalized definition of the concurrence for qutrit states.

Suppose that the initial pure state is $\rho(0) = |\omega\rangle\langle\omega|$, where $|\omega\rangle = a|02\rangle + b|11\rangle + c|20\rangle$ where the coefficients a, b , and c satisfy $\sqrt{a^2 + b^2 + c^2} = 1$. It is very difficult to discuss an arbitrary case. Instead, we focus on the following special cases.

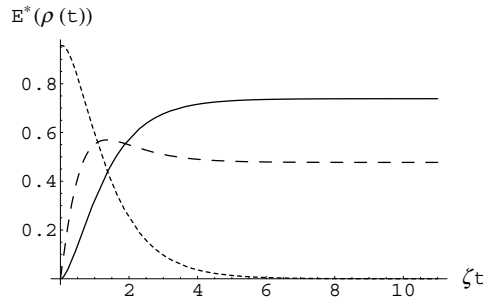


Figure 7. The upper bound E^* of the entanglement of two-photon process for different cases: $a = 1, b = c = 0$ (solid line), $c = 1, a = b = 0$ (dashed line), and $a = b = \frac{1}{2}, c = 1/\sqrt{2}$ (dotted line).

Case 1, $a = 1, b = c = 0$. The time-dependent solution can be obtained as

$$\begin{aligned} \rho(t) &= \frac{(e^{-\zeta t} + 1)^2}{4} |\varphi(t)\rangle\langle\varphi(t)| + \frac{1 - e^{-2\zeta t}}{2} |\phi(t)\rangle\langle\phi(t)| + \frac{(1 - e^{-\zeta t})^2}{4} |00\rangle\langle 00|, \\ |\varphi(t)\rangle &= \frac{(1 + e^{-\zeta t/2})^2}{2(e^{-\zeta t} + 1)} |02\rangle - \frac{1 - e^{-\zeta t}}{\sqrt{2}(e^{-\zeta t} + 1)} |11\rangle + \frac{(1 - e^{-\zeta t/2})^2}{2(e^{-\zeta t} + 1)} |20\rangle, \\ |\phi(t)\rangle &= \frac{1 + e^{-\zeta t/2}}{\sqrt{2}(e^{-\zeta t} + 1)} |01\rangle - \frac{1 - e^{-\zeta t/2}}{\sqrt{2}(e^{-\zeta t} + 1)} |10\rangle. \end{aligned} \tag{25}$$

From equations (25) we can see that with the time evolution $\rho(t)$ tends to $\frac{1}{4}|\varphi\rangle\langle\varphi| + \frac{1}{2}|\phi\rangle\langle\phi| + \frac{1}{4}|00\rangle\langle 00|$, which is a mixture of two-photon, one-photon and zero-photon zero-mode solutions, as analysed in section 4.1. Since the quantification of entanglement of mixed entangled qutrit states is not yet well understood, we can give an upper bound of the entanglement by the convexity property of entropy function, namely,

$$\begin{aligned} E(\rho(t)) &\leq \frac{(e^{-\zeta t} + 1)^2}{4} E(|\varphi(t)\rangle\langle\varphi(t)|) + \frac{1 - e^{-2\zeta t}}{2} E(|\phi(t)\rangle\langle\phi(t)|) \\ &\quad + \frac{(1 - e^{-\zeta t})^2}{4} E(|00\rangle\langle 00|) = E^*(\rho(t)). \end{aligned}$$

In the above expression, E is defined by $E = h\left(\frac{1+\sqrt{1-C^2}}{2}\right)$, where $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ and C is the concurrence of the related density matrix. From equation (24) the two-photon pure state entanglement $E(|\varphi(t)\rangle\langle\varphi(t)|)$ can be obtained. $E(|\phi(t)\rangle\langle\phi(t)|)$ is just a pure qubit entanglement, which can be calculated in the traditional way. Figure 7 shows the time evolution of $E^*(\rho(t))$ (solid line). One notes that the steady entanglement can be generated from the product initial state.

The similar discussions can also be applied to the cases of $c = 1, a = b = 0$ and $b = 1, a = c = 0$. While the evolution of the upper bound $E^*(\rho(t))$ for $b = 1, a = c = 0$ is the same as the above, a slightly different evolution in the case of $c = 1, a = b = 0$ is found, as shown in figure 7 by the dashed line. Nevertheless, the conclusion of the generation of the entanglement is same.

Case 2, $a = b = \frac{1}{2}, c = -1/\sqrt{2}$. In this case the solution is time-independent,

$$\rho(t) = |\varphi\rangle\langle\varphi|,$$

where $|\varphi\rangle$ is the same as the form in section 4.1. Once again we verify the existence of the decoherence-free state.

Case 3, $a = b = \frac{1}{2}$, $c = 1/\sqrt{2}$. From the analysis of the last subsection we know that environment can produce and destroy entanglement, which is dependent on the initial state. This is still true in the two-photon process. Case 1 corresponds to the entanglement production. In this case we shall see the environment can destroy entanglement in certain initial states. The solution read

$$\begin{aligned}\rho(t) &= e^{-2\zeta t} |\chi\rangle\langle\chi| + 2(e^{-\zeta t} - e^{-2\zeta t}) |v\rangle\langle v| + (1 - e^{-\zeta t})^2 |00\rangle\langle 00|, \\ |\chi\rangle &= \frac{1}{2}|02\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{2}|20\rangle, \\ |v\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle.\end{aligned}$$

We can see $\rho(t)$ tends to zero-photon zero-mode solution $|00\rangle\langle 00|$. As a result, the entanglement is destroyed completely. The time-evolution for $E^*(\rho(t))$ is plotted in figure 7, as shown as dotted line.

The discussion above is limited to very special initial states, which are formulated in the Fock states representation. It is very interesting to expand the present discussion to the Gaussian states, which can be created relatively easily and can be widely applied to quantum cryptography and quantum teleportation. In addition, the entanglement criteria [28–30] and the formation of entanglement [31] are also available. However, instead of the Fock state representation, it is very convenient to use the continuous variable representation to discuss the Gaussian states, which is beyond the present work.

6. Summary

In summary, we have investigated the master equation of two identical single-mode cavity fields coupled to a common quantum environment. Using the algebraic dynamical method, the two kinds of dynamical symmetries, namely the $sp(4) \supset su(1, 1) \supset \{K_0\}$ and $sp(4) \supset u(2) \supset \{U, S_0\}$ dynamical symmetries of the master equation have been explored. The first dynamical symmetry corresponds to the collective quasi-photon mode, while the second one is the dynamical symmetry for the real photons of the two cavities. According to the algebraic dynamics, the system is integrable and its analytical solutions have been obtained. It is found that the system is affected by the environment through the rate operator consisting merely of collective quasi-photon operators and having collective quasi-photon eigenmodes which are highly degenerate in terms of the real photon states. Based on the analytical solutions, the decoherence and entanglement properties of the quasi-photon zero mode of the rate operator have been studied in detail, and the surprising finding is that different entangled states can be produced from the disentangled initial states by environment effects in some cases. The results obtained are useful for the encoding scheme in quantum computation and for the entanglement generation of the 2-qubit and 2-qutrit cavity systems.

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